

The Recent Excitement in High-Density QCD *

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Over the past few months, the theory of QCD at high density has been advanced considerably. It provides new perspectives on, and controlled realizations of, confinement and chiral symmetry breaking. Here I survey the recent developments, and suggest a few directions for future work.

1. Introduction

The behavior of QCD at high density is intrinsically interesting, as the answer to the question: What happens to matter, if you keep squeezing it harder and harder? It is also directly relevant to the description of neutron star interiors, neutron star collisions, and events near the core of collapsing stars. Also, one might hope to obtain some insight into physics at “low” density – that is, ordinary nuclear density or just above – by approaching it from the high-density side.

Why might we anticipate QCD simplifies in the limit of high density? A crude answer is: “Asymptotic freedom meets the fermi surface.” One might argue, formally, that the only external mass scale characterizing the problem is the large chemical potential μ , so that if the effective coupling $\alpha_s(\mu)$ is small, as it will be for $\mu \gg \Lambda_{QCD}$, where $\Lambda_{QCD} \approx 200$ Mev is the primary QCD scale, then we have a weak coupling problem. More physically, one might argue that at large μ the relevant, low-energy degrees of freedom involve modes near the fermi surface, which have large energy and momentum. An interaction between particles in these modes will either barely deflect them, or will involve a large momentum transfer. In the first case we don’t care, while the second is governed by a small effective coupling.

These arguments are too quick, however. The formal argument is specious, if the perturbative expansion contains infrared divergences. And there are good reasons – two separate ones, in fact – to anticipate such divergences.

First, fermi balls are generically unstable against the effect of attractive interactions, however weak, between pairs near the fermi surface that carry equal and opposite momentum. This is the Cooper instability, which drives ordinary superconductivity in metals and the superfluidity of He3. It is possible because occupied pair states can have very low energy, and they can all scatter into one another. Thus one is doing highly degenerate perturbation theory, and in such a situation even a very weak coupling can produce

*Invited talk at PANIC '99, Uppsala, Sweden, June 1999. IASSNS-HEP/99-68

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significant “nonperturbative” effects.

Second, nothing in our heuristic argument touches the gluons. To be sure the gluons will be subject to electric screening, but at zero frequency there is no magnetic screening, and infrared divergences do in fact arise, through exchange of soft magnetic gluons.

Fortunately, by persisting along this line of thought we find a path through the apparent difficulties. Several decades ago Bardeen, Cooper, and Schrieffer taught us, in the context of metallic superconductors, how the Cooper instability is resolved [1]. We can easily adapt their methods to QCD [2] [3]. In electronic systems only rather subtle mechanisms can generate an attractive effective interaction near the fermi surface, since the primary electron-electron interaction is Coulomb repulsion. In QCD, remarkably, it is much more straightforward. Even at the crudest level we find attraction. Indeed, two quarks, each a color triplet, can combine to form a single color antitriplet, thus reducing their total field energy.

The true ground state of the quarks is quite different from the naive fermi balls. It is characterized by the formation of a coherent condensate, and the development of an energy gap. The condensation, which is energetically favorable, is inconsistent with a magnetic field, and so weak magnetic fields are expelled. This is the famous Meissner effect in superconductivity, which is essentially identical to what is known as the Higgs phenomenon in particle physics. Magnetic screening of gluons, together with energy gaps for quark excitations, remove the potential sources of infrared divergences mentioned above. Thus we have good reasons to hope that a weak coupling – though, of course, nonperturbative – treatment of the high density state will be fully consistent and accurate.

The central result in the recent developments is that this program can be carried to completion rigorously in QCD with sufficiently many (three or more) quark species [4]. Thus the more refined, and fully adequate, answer to our earlier question is: “Asymptotic freedom meets the BCS groundstate.” Together, these concepts render the behavior of QCD at asymptotically high density calculable.

The simplest and most beautiful results, luckily, occur in the version of QCD containing three quarks having equal masses. I say luckily, because this idealization applies to the real world, at densities so high that we can neglect the strange quark mass (yet not so high that we have to worry about charmed quarks). Here we encounter the phenomenon of color-flavor locking. The ground state contains correlations whereby both color and flavor symmetry are spontaneously broken, but the diagonal subgroup, which applies both transformations simultaneously, remains valid.

Color-flavor locking has many remarkable consequences [4]. There is a gap for all colored excitations, including the gluons. This is, operationally, confinement. The photon picks up a gluonic component of just such a form as to ensure that all elementary excitations, including quarks, are integrally charged. Some of the gluons acquire non-zero, but integer-valued, electric charges. Baryon number is spontaneously broken, which renders the high-density material a superfluid.

If in addition the quarks are massless, then their chiral symmetry is spontaneously broken, by a new mechanism. The left-dynamics and the right-dynamics separately lock to color; but since color allows only vector transformations, left is thereby locked to right.

You may notice several points of resemblance between the low-energy properties calculated for the high-density color-flavor locked phase and the ones you might expect at low

density, based on semi-phenomenological considerations such as the MIT bag model, or experimental results in real-world QCD. The quarks play the role of low-lying baryons, the gluons play the role of the low-lying vector mesons, and the Nambu-Goldstone bosons of broken chiral symmetry play the role of the pseudoscalar octet. All the quantum numbers match, and the spectrum has gaps – or not – in all the right places. In addition we have baryon number superfluidity, which extends the expected pairing phenomena in nuclei. Overall, there is an uncanny match between all the universal, and several of the non-universal, features of the calculable high-density and the expected low-density phase. This leads us to suspect that there is no phase transition between them [5]!

2. Sculpting the Problem

2.1. renormalization group toward the fermi surface

To sculpt the problem, begin by assuming weak coupling, and focus on the quarks. Then the starting point is fermi balls for all the quarks, and the low-energy excitations include states where some modes below the nominal fermi surface are vacant and some modes above are occupied. The renormalization group, in a generalized sense, is a philosophy for dealing with problems involving nearly degenerate perturbation theory. In this approach, one attempts to map the original problem onto a problem with fewer degrees of freedom, by integrating out the effect of the higher-energy (or, in a relativistic theory, more virtual) modes. Then one finds a new formulation of the problem, in a smaller space, with new couplings. In favorable cases the reformulated problem is simpler than the original, and one can go ahead and solve it.

This account of the renormalization group might seem odd, at first sight, to high-energy physicists accustomed to using asymptotic freedom in QCD. That is because in traditional perturbative QCD one runs the procedure backward. When one integrates out highly virtual modes, one finds the theory becomes more strongly coupled. Simplicity arises when one asks questions that are somehow inclusive, so that to answer them one need not integrate out very much. It is then that the microscopic theory, which is ideally symmetric and constrained, applies directly. So one might say that the usual application of the renormalization group in QCD is fundamentally negative: it informs us how the fundamentally simple theory comes to look complicated at low energy, and helps us to identify situations where we can avoid the complexity.

Here, although we are still dealing with QCD, we are invoking quite a different renormalization group, one which conforms more closely to the Wilsonian paradigm [6] [7]. We consider the effect of integrating out modes whose energy is within the band $(\epsilon, \delta\epsilon)$ of the fermi surface, on the modes of lower energy. This will renormalize the couplings of the remaining modes, due to graphs like those displayed in Figure 1. In addition the effect of higher-point interactions is suppressed, because the phase space for them shrinks, and it turns out that only four-fermion couplings survive unscathed (they are the marginal, as opposed to irrelevant, interactions). Indeed the most significant interactions are those involving particles or holes with equal and opposite three momenta, since they can scatter through many intermediate states. For couplings g_η of this kind we find

$$\frac{dg_\eta}{d \ln \delta} = \kappa_\eta g_\eta^2. \tag{1}$$

Here η labels the color, flavor, angular momentum, ... channel and in general we have a matrix equation [8] [9] – but let's keep it simple, so κ_η is a positive number. Then 1 is quite simple to integrate, and we have

$$\frac{1}{g_\eta(1)} - \frac{1}{g_\eta(\delta)} = \kappa_\eta \ln \delta. \quad (2)$$

Thus for $g_\eta(1)$ negative, corresponding to attraction, $|g_\eta(\delta)|$ will grow as $\delta \rightarrow 0$, and become singular when

$$\delta = e^{\frac{1}{\kappa_\eta g_\eta(1)}}. \quad (3)$$

Note that although the singularity occurs for arbitrarily weak attractive coupling, it is nonperturbative.

2.2. model hamiltonian and condensation

The renormalization group toward the fermi surface helps us identify potential instabilities, but it does not indicate how they are resolved. The great achievement of BCS was to identify the form of the stable ground state the Cooper instability leads to. Their original calculation was variational, and that is still the most profound and informative approach, but simpler, operationally equivalent algorithms are now more commonly used. I will be very sketchy here, since this is textbook material.

Most calculations to date have been based on model interaction Hamiltonians, that are motivated, but not strictly derived, from microscopic QCD. They are chosen as a compromise between realism and tractability. For concreteness I shall here follow [4], and consider

$$\begin{aligned} H &= \int d^3x \bar{\psi}(x)(\nabla - \mu\gamma_0)\psi(x) + H_I, \\ H_I &= K \sum_{\mu,A} \int d^3x \mathcal{F} \bar{\psi}(x) \gamma_\mu T^A \psi(x) \bar{\psi}(x) \gamma^\mu T^A \psi(x) \end{aligned} \quad (4)$$

Here the T^A are the color $SU(3)$ generators, so the quantum numbers are those of one-gluon exchange. However instead of an honest gluon propagator we use an instantaneous contact interaction, modified by a form-factor \mathcal{F} . \mathcal{F} is taken to be a product of several momentum dependent factors $F(p)$, one for each leg, and to die off at large momentum. One convenient possibility is $F(p) = (\lambda^2/(p^2 + \lambda^2))^\nu$, where λ and ν can be varied to study sensitivity to the location and shape of the cutoff. The qualitative effect of the form-factor is to damp the spurious ultraviolet singularities introduced by H_I ; microscopic QCD, of course, does have good ultraviolet behavior. One will tend to trust conclusions that do not depend sensitively on λ or ν . In practice, one finds that the crucial results – the form and magnitude of gaps – are rather forgiving.

Given the Hamiltonian, we can study the possibilities for symmetry breaking condensations. The most favorable condensation possibility so far identified is of the form

$$\langle q_{La}^{i\alpha}(p) q_{Lb}^{j\beta}(-p) \rangle = - \langle q_{Ra}^{i\alpha}(p) q_{Rb}^{j\beta}(-p) \rangle = \epsilon^{ij} (\kappa_1(p^2) \delta_a^\alpha \delta_b^\beta + \kappa_2(p^2) \delta_b^\alpha \delta_a^\beta). \quad (5)$$

There are several good reasons to think that condensation of this form characterizes the true ground state, with lowest energy, at asymptotic densities. It corresponds to the most

singular channel, in the renormalization group analysis discussed above. It produces a gap in all channels, and is perturbatively stable, so that it is certainly a convincing local minimum. And it beats various more-or-less plausible competitors that have been investigated, by a wide margin.

Given the form of the condensate, one can fix the leading functional dependencies of $\kappa_1(p^2, \mu)$ and $\kappa_2(p^2, \mu)$ at weak coupling by a variational calculation. For present purposes, it is adequate to replace all possible contractions of the quark fields in 4 having the quantum numbers of 5 with their supposed expectation values, and diagonalize the quadratic part of the resulting Hamiltonian. The ground state is obtained, of course, by filling the lowest energy modes, up to the desired density. One then demands internal consistency, *i.e.* that the postulated expectation values are equal to the derived ones. Some tricky but basically straightforward algebra leads us to the result

$$\Delta_{1,8}(p^2) = F(p)^2 \Delta_{1,8} \quad (6)$$

where Δ_1 and Δ_8 satisfy the coupled gap equations

$$\begin{aligned} \Delta_8 + \frac{1}{4}\Delta_1 &= \frac{16}{3}KG(\Delta_1) \\ \frac{1}{8}\Delta_1 &= \frac{16}{3}KG(\Delta_8) \end{aligned} \quad (7)$$

where we have defined

$$G(\Delta) = -\frac{1}{2} \sum_{\mathbf{k}} \left\{ \frac{F(k)^4 \Delta}{\sqrt{(k-\mu)^2 + F(k)^4 \Delta^2}} + \frac{F(k)^4 \Delta}{\sqrt{(k+\mu)^2 + F(k)^4 \Delta^2}} \right\} \quad (8)$$

and

$$\begin{aligned} \kappa_1(p^2) &= \frac{1}{8K}(\Delta_8(p^2) + \frac{1}{8}\Delta_1(p^2)) \\ \kappa_2(p^2) &= \frac{3}{64K}\Delta_1(p^2). \end{aligned} \quad (9)$$

The Δ are defined so that $F(p)^2 \Delta_{1,8}(p^2)$ are the gaps for singlet or octet excitations at 3-momentum p . Equations 8 must be solved numerically.

Finally, to obtain quantitative estimates of the gaps, we must normalize the parameters of our model Hamiltonian. One can do this very crudely by using the model Hamiltonian in the manner originally pioneered by Nambu and Jona-Lasinio [10], that is as the basis for a variational calculation of chiral symmetry breaking at zero density. The magnitude of this chiral condensate can then be fixed to experimental or numerical results. In this application we have no firm connection between the model and microscopic QCD, because there is no large momentum scale (or weak coupling parameter) in sight. Nevertheless a very large literature following this approach encourages us to hope that its results are not wildly wrong, quantitatively. Upon adopting this normalization procedure, one finds that gaps of order several tens of Mev near the fermi surface are possible at moderate densities.

While this model treatment captures major features of the physics of color-flavor locking, with a little more work it is possible to do a much more rigorous calculation, and in particular to normalize directly to the known running of the coupling at large momentum. This will be sketched below.

3. Consequences of Color-Flavor Locking: Symmetry

3.1. broken gauge invariance?

An aspect of 5 that might appear troubling at first sight, is its lack of gauge invariance. There are powerful general arguments that local gauge invariance cannot be broken [11]. Indeed, local gauge invariance is really a tautology, stating the equality between redundant variables. Yet its ‘breaking’ is central to two of the most successful theories in physics, to wit BCS superconductivity theory and the standard model of electroweak interactions. In BCS theory we postulate a non-zero vacuum expectation value for the (electrically charged) Cooper pair field, and in the standard model we postulate a non-zero vacuum expectation value for the Higgs field, which violates both the weak isospin $SU(2)$ and the weak hypercharge $U(1)$.

In each case, we should interpret the condensate as follows. We are working in a gauge theory at weak coupling. It is then very convenient to fix a gauge, because after we have done so – but not before! – the gauge potentials will make only small fluctuations around zero, which we will be able to take into account perturbatively. Of course at the end of any calculation we must restore the gauge symmetry, by averaging over the gauge fixing parameters (gauge unfixing). Only gauge-invariant results will survive this averaging. In a fixed gauge, however, one might capture important correlations, that characterize the ground state, by specifying the existence of non-zero condensates relative to that gauge choice. These condensates need not, and generally will not, break any symmetries.

For example, in the standard electroweak model one employs a non-zero vacuum expectation value for a Higgs doublet field $\langle \phi^a \rangle = v\delta_1^a$, which is not gauge invariant. One might be tempted to use the magnitude of its absolute square, which is gauge invariant, as an order parameter for the symmetry breaking, but $\langle \phi^\dagger \phi \rangle$ never vanishes, whether or not any symmetry is broken (and, of course, $\langle \phi^\dagger \phi \rangle$ breaks no symmetry). In fact there is no order parameter for the electroweak phase transition, and it has long been appreciated [12] that one could, by allowing the $SU(2)$ gauge couplings to become large, go over into a ‘confined’ regime while encountering no sharp phase transition. The most important gauge-invariant consequences one ordinarily infers from the condensate, of course, are the non-vanishing W and Z boson masses. This absence of massless bosons and long-range forces is the essence of confinement, or of the Meissner-Higgs effect. Evidently, when used with care, the notion of spontaneous gauge symmetry breaking can be an extremely convenient fiction – so it proves for 5.

3.2. broken symmetries and consequences

The equations of our original model, QCD with three massless flavors, has the continuous symmetry group $SU(3)^c \times SU(3)_L \times SU(3)_R \times U(1)_B$. The Kronecker deltas that appear in the condensate 5 are invariant under neither color nor left-handed flavor nor right-handed flavor rotations separately. Only a global, diagonal $SU(3)$ leaves the ground state invariant. Thus we have the symmetry breaking pattern

$$SU(3)^c \times SU(3)_L \times SU(3)_R \times U(1)_B \rightarrow SU(3)_{c+L+R} \times Z_2 . \quad (10)$$

Indeed, if we make a left-handed chiral rotation we can compensate it by a color rotation, to leave the left-handed condensate invariant. Color rotations being vectorial, we

must then in addition make a right-handed chiral rotation, in order to leave the right-handed condensate invariant. Thus chiral symmetry is spontaneously broken, by a new mechanism: although the left- and right- condensates are quite separate (and, before we include instantons – see below – not even phase coherent), because both are locked to color they are thereby locked to one another.

The breaking of local color symmetry implies that all the gluons acquire mass, according to the Meissner (or alternatively Higgs) effect. There are no long-range, $1/r$ interactions. There is no direct signature for the color degree of freedom – although, of course, in weak coupling one clearly perceives its avatars. It is veiled or, if you like, confined.

The spontaneous breaking of global chiral $SU(3)_L \times SU(3)_R$ brings with it an octet of pseudoscalar Nambu-Goldstone bosons, collective modes interpolating, in space-time, among the condensates related by the lost symmetry. These massless modes, as is familiar, are derivatively coupled, and therefore they do not generate singular long-range interactions.

Less familiar, and perhaps disconcerting at first sight, is the loss of baryon number symmetry. This does not, however, portend proton decay, any more than does the non-vanishing condensate of helium atoms in superfluid He4. Given an isolated finite sample, the current divergence equation can be integrated over a surface surrounding the sample, and unambiguously indicates overall number conservation. To respect it, one should project onto states with a definite number of baryons, by integrating over states with different values of the condensate phase. This does not substantially alter the physics of the condensate, however, because the overlap between states of different phase is very small for a macroscopic sample. Roughly speaking, there is a finite mismatch per unit volume, so the overlap vanishes exponentially in the limit of infinite volume. The true meaning of the formal baryon number violation is that there are low-energy states with different distributions of baryon number, and easy transport among them. Indeed, the dynamics of the condensate is the dynamics of superfluidity: gradients in the Nambu-Goldstone mode are none other than the superfluid flow.

We know from experience that large nuclei exhibit strong even-odd effects, and an extensive phenomenology has been built up around the idea of pairing in nuclei. If electromagnetic Coulomb forces didn't spoil the fun, we could confidently expect that extended nuclear matter would exhibit the classic signatures of superfluidity. In our 3-flavor version the Coulomb forces do not come powerfully into play, since the charges of the quarks average out to electric neutrality. Furthermore, the tendency to superfluidity exhibited by ordinary nuclear matter should be enhanced by the additional channels operating coherently. So one should expect strong superfluidity at ordinary nuclear density, and it becomes less surprising that we find it at asymptotically large density too.

3.3. true order parameters

I mentioned before that the Higgs mechanism as it operates in the electroweak sector of the standard model has no gauge-invariant signature. With color-flavor locking we're in better shape, because global as well as gauge symmetries are broken. Thus there are sharp differences between the color-flavor locked phase and the free phase. There must be phase transitions – as a function, say, of temperature – separating them.

In fact, it is a simple matter to extract gauge invariant order parameters from our

primary, gauge variant condensate at weak coupling. For instance, to form a gauge invariant order parameter capturing chiral symmetry breaking we may take the product of the left-handed version of 5 with the right-handed version and saturate the color indices, to obtain

$$\langle q_{La}^\alpha q_{Lb}^\beta \bar{q}_{R\alpha}^c \bar{q}_{R\beta}^d \rangle \sim \langle q_{La}^\alpha q_{Lb}^\beta \rangle \langle \bar{q}_{R\alpha}^c \bar{q}_{R\beta}^d \rangle \sim (\kappa_1^2 + \kappa_2^2) \delta_a^c \delta_b^d + 2\kappa_1 \kappa_2 \delta_a^d \delta_b^c \quad (11)$$

Likewise we can take a product of three copies of the condensate and saturate the color indices, to obtain a gauge invariant order parameter for superfluidity. These secondary order parameters will survive gauge unfixing unscathed. Unlike the primary condensate from which they were derived, they are more than convenient fictions.

3.4. a subtlety: axial baryon number

As it stands the chiral order parameter 11 is not quite the usual one, but roughly speaking its square. It leaves invariant an additional Z_2 , under which the left-handed quark fields change sign. Actually this Z_2 is not a legitimate symmetry of the full theory, but suffers from an anomaly.

Since we can be working at weak coupling, we can be more specific. Our model Hamiltonian 4 was abstracted from one-gluon exchange, which is the main interaction among high-energy quarks in general, and so in particular for modes near our large fermi surfaces. The instanton interaction is much less important, at least asymptotically, both because it is intrinsically smaller for energetic quarks, and because it involves six fermion fields, and hence (one can show) is irrelevant as one renormalizes toward the fermi surface. However, it represents the leading contribution to axial baryon number violation. In particular, it is only $U_A(1)$ violating interactions that fix the relative phase of our left- and right-handed condensates. So a model Hamiltonian that neglects them will have an additional symmetry that is not present in the full theory. After spontaneous breaking, which does occur in the axial baryon number channel, there will be a Nambu-Goldstone boson in the model theory, that in the full theory acquires an anomalously (pun intended) small mass [13]. Similarly, in the full theory there will be a non-zero tertiary chiral condensate of the usual kind, bilinear in quark fields, but it will be parametrically smaller than 11.

4. Consequences of Color-Flavor Locking: Elementary Excitations

There are three sorts of elementary excitations. They are the modes produced directly by the fundamental quark and gluon fields, and the collective modes connected with spontaneous symmetry breaking.

The quark fields of course produce spin 1/2 fermions. Some of these are true long-lived quasiparticles, since there is nothing for them to decay into. They form an octet and a singlet under the residual diagonal $SU(3)$. There is an energy gap for production of pairs above the ground state. Actually there are two gaps: a smaller one for the octet, and a larger one for the singlet.

The gluon fields produce an octet of spin 1 bosons. As previously mentioned, they acquire a mass by the Meissner-Higgs phenomenon. We have already discussed the Nambu-Goldstone bosons, too.

4.1. modified photon and integer charges

The notion of ‘confinement’ I advertised earlier, phrased in terms of mass gaps and derivative interactions, might seem rather disembodied. So it is interesting to ask whether and how a more traditional and intuitive criterion of confinement – no fractionally charged excitations – is satisfied.

Before discussing electromagnetic charge we must identify the unbroken gauge symmetry, whose gauge boson defines the physical photon in our dense medium. The original electromagnetic gauge invariance is broken, but there is a combination of the original electromagnetic gauge symmetry and a color transformation which leaves the condensate invariant. Specifically, the original photon γ couples according to the matrix

$$\begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \quad (12)$$

in flavor space, with strength e . There is a gluon G which couples to the matrix

$$\begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \quad (13)$$

in color space, with strength g . Then the combination

$$\tilde{\gamma} = \frac{g\gamma + eG}{\sqrt{e^2 + g^2}} \quad (14)$$

leaves the ‘locking’ Kronecker deltas in color-flavor space invariant. In our medium, it represents the physical photon. What happens here is similar to what occurs in the electroweak sector of the standard model, where both weak isospin and weak hypercharge are separately broken by the Higgs doublet, but a cunning combination remains unbroken, and defines electromagnetism.

Now with respect to $\tilde{\gamma}$ the electron charge is

$$\frac{-eg}{\sqrt{e^2 + g^2}}, \quad (15)$$

deriving of course solely from the γ piece of 14. The quarks have one flavor and one color index, so they pick up contributions from both pieces. In each sector we find the normalized charge unit $\frac{eg}{\sqrt{e^2 + g^2}}$, and it is multiplied by some choice from among $(2/3, -1/3, -1/3)$ or $(-2/3, 1/3, 1/3)$ respectively. The total, obviously, can be ± 1 or 0. Thus the excitations produced by the quark fields are integrally charged, in units of the electron charge. Similarly the gluons have an upper color and a lower anti-color index, so that one faces similar choices, and reaches a similar conclusion. In particular, some of the gluons have become electrically charged. The pseudoscalar Nambu-Goldstone modes have an upper flavor and a lower anti-flavor index, and yet again the same conclusions follow. The superfluid mode, of course, is electrically neutral.

It is fun to consider how a chunk of our color-flavor locked material would look. If the quarks were truly massless, then so would be Nambu-Goldstone bosons (at the level

of pure QCD), and one might expect a rather unusual ‘bosonic metal’, in which low-energy electromagnetic response is dominated by these modes. Actually electromagnetic radiative corrections lift the mass of the charged Nambu-Goldstone bosons, creating a gap for the charged channel. The same effect would be achieved by turning on a common non-zero quark mass. Thus the color-flavor locked material forms a transparent insulator. Altogether it resembles a diamond, that reflects portions of incident light waves, but allows finite portions through and out again!

4.2. quark-hadron continuity

The universal features of the color-flavor locked state: confinement, chiral symmetry breaking down to vector $SU(3)$, and superfluidity, are just what one would expect, based on standard phenomenological models and experience with real-world QCD at low density. Now we see that the low-lying spectrum likewise bears an uncanny resemblance to what one finds in the Particle Data Book (or rather what one would find, in a world of three degenerate quarks). It is hard to resist the inference that there is no phase transition separating them. Thus there need not be, and presumably is not, a sharp distinction between the low-density phase, where microscopic calculations are difficult but the convenient degrees of freedom are “obviously” hadrons, and the asymptotic high-density phase, where weak-coupling (but non-perturbative) calculations are possible, and the right degrees of freedom are elementary quarks and gluons plus collective modes associated with spontaneous symmetry breaking. We call this quark-hadron continuity [5]. It might seem shocking that a quark can “be” a baryon, but remember that it is immersed in a sea of diquark condensate, wherein the distinction between one quark and three is negotiable.

4.3. remembrance of things past

An entertaining aspect of the emergent structure is that two beautiful ideas from the pre-history of QCD, that were bypassed in its later development, have come very much back to center stage, now with microscopic validation. The quark-baryons of the color-flavor locked phase follow the charge assignments proposed by Han and Nambu [14]. And the gluon-vector mesons derive from the Yang-Mills gauge principle [15] – as originally proposed, for rho mesons!

5. Fully Microscopic Calculation

A proper discussion of the fully microscopic calculation [16] [17] [18] [19] is necessarily quite technical, and would be out of place here, but the spirit of the thing – and one of the most striking results – can be conveyed simply.

When retardation or relativistic effects are important a Hamiltonian treatment is no longer appropriate. One must pass to Lagrangian and graphical methods. (Theoretical challenge: is it possible to systematize these in a variational approach?) The gap equation appears as a self-consistency equation for the assumed condensation, shown graphically in Figure 2.

With a contact interaction, and throwing away manifestly spurious ultraviolet divergences, we obtain a gap equation of the type

$$\Delta \propto g^2 \int d\epsilon \frac{\Delta}{\sqrt{\epsilon^2 + \Delta^2}}. \quad (16)$$

The phase space transverse to the fermi surface cancels against a propagator, leaving the integral over the longitudinal distance ϵ to the fermi surface. Note that the integral on the right diverges at small ϵ , so that as long as the proportionality constant is positive one will have non-trivial solutions for Δ , no matter how small is g . Indeed, one finds that for small g , $\Delta \sim e^{-\text{const}/g^2}$.

If we restore the gluon propagator, we will find a non-trivial angular integral, which diverges for forward scattering. That divergence will be killed, however, if the gluon acquires a mass $\propto g\Delta$ through the Meissner-Higgs mechanism. Thus we arrive at a gap equation of the type

$$\Delta \propto g^2 \int d\epsilon \frac{\Delta}{\sqrt{\epsilon^2 + \Delta^2}} dz \frac{\mu^2}{\mu^2 + (g\Delta)^2}. \quad (17)$$

Now one finds $\Delta \sim e^{-\text{const}/g!}$

A proper discussion of the microscopic gap equation is considerably more involved than this, but the conclusion that the gap goes exponentially in the inverse coupling (rather than its square) at weak coupling still emerges. It has the amusing consequence, that at asymptotically high densities the gap becomes arbitrarily large! This is because asymptotic freedom insures that it is the microscopic coupling $1/g(\mu)^2$ which vanishes logarithmically, so that $e^{-\text{const}/g(\mu)}$ does not shrink as fast as $1/\mu$. Since the ‘‘dimensional analysis’’ scale of the gap is set by μ , its linear growth wins out asymptotically.

6. More Quarks

For larger numbers of quarks, the story is qualitatively similar [5]. Color symmetry is broken completely, and there is a gap in all quark channels, so the weak-coupling treatment is adequate. Color-flavor locking is so favorable that there seems to be a periodicity: if the number of quarks is a multiple of three, one finds condensation into 3×3 blocks, while if it is $4+3k$ or $5+3k$ one finds k color-flavor locking blocks together with special patterns characteristic of 4 or 5 flavors.

There is an amusing point here. QCD with a very large number of massless quarks, say 16, has an infrared fixed point at very weak coupling [20] [21]. Thus it should be quasi-free at zero density, forming a nonabelian Coulomb phase, featuring conformal symmetry, no confinement, and no chiral symmetry breaking. To say the least, it does not much resemble real-world QCD. There are indications that this qualitative behavior may persist even for considerably fewer quarks (the critical number might be as small as 5 or 6). Nevertheless, at high density, we have discovered, these many-quark theories all support more-or-less normal-looking ‘nuclear matter’ – including confinement and chiral symmetry breaking!

7. Fewer Quarks

7.1. two flavors

One can perform a similar analysis for two quark flavors [22] [23]. A new feature is that the instanton interaction now involves four rather than six quark legs, so it remains relevant as one renormalizes toward the fermi surface. Either the one-gluon exchange or the instanton interaction, treated in the spirit above, favors condensation of the form

$$\langle q_{La}^{i\alpha}(p) q_{Lb}^{j\beta}(-p) \rangle = - \langle q_{Ra}^{i\alpha}(p) q_{Lb}^{j\beta}(-p) \rangle = \epsilon^{ij} \kappa(p^2) \epsilon^{\alpha\beta 3} \epsilon_{ab} . \quad (18)$$

Formally, 18 is quite closely related to 5, since $\epsilon^{\alpha\beta I}\epsilon_{abI} = 2(\delta_a^\alpha\delta_b^\beta - \delta_b^\alpha\delta_a^\beta)$. Their physical implications, however, are quite different.

To begin with, 18 does not lead to gaps in all quark channels. The quarks with color labels 1 and 2 acquire a gap, but quarks of the third color of quark are left untouched. Secondly, the color symmetry is not completely broken. A residual $SU(2)$, acting among the first two colors, remains valid. For these reasons, perturbation theory about the ground state defined by 18 is *not* free of infrared divergences, and we do not have a fully reliable grip on the physics.

Nevertheless it is plausible that the qualitative features suggested by 18 are not grossly misleading. The residual $SU(2)$ presumably produces confined glueballs of large mass, and assuming this occurs, the residual gapless quarks are weakly coupled.

Assuming for the moment that no further condensation occurs, for massless quarks we have the symmetry breaking pattern

$$SU(3)^c \times SU(2)_L \times SU(2)_R \times U(1)_B \rightarrow SU(2)^c \times SU(2)_L \times SU(2)_R \times \tilde{U}(1)_B \quad (19)$$

Here the modified baryon number acts only on the third color of quarks. It is a combination of the original baryon number and a color generator, that are separately broken but when applied together leave the condensate invariant. Comparing to the zero-density ground state, one sees that color symmetry is reduced, chiral symmetry is restored, and baryon number is modified. Only the restoration of chiral symmetry is associated with a legitimate order parameter, and only it requires a sharp phase transition.

In the real world, with the u and d quarks light but not strictly massless, there is no rigorous argument that a phase transition is necessary. It is (barely) conceivable that one might extend quark-hadron continuity to this case [24]. Due to medium modifications of baryon number and electromagnetic charge the third-color u and d quarks have the quantum numbers of nucleons. The idea that chiral symmetry is effectively restored in nuclear matter, however, seems problematic quantitatively. More plausible, perhaps, is that there is a first-order transition between nuclear matter and quark matter. This is suggested by some model calculations (e.g., [22] [23]), and is the basis for an attractive interpretation of the MIT bag model, according to which baryons are droplets wherein chiral symmetry is restored.

7.2. thresholds and mismatches

In the real world there are two quarks, u and d , whose mass is much less than Λ_{QCD} , and one, s , whose mass is comparable to it. Two simple qualitative effects, that have major implications for the zero-temperature phase diagram, arise as consequences of this asymmetric spectrum [24] [25]. They are expected, whether one analyzes from the quark side or from the hadron side.

The first is that one can expect a threshold, in chemical potential (or pressure), for the appearance of any strangeness at all in the ground state. This will certainly hold true in the limit of large strange quark mass, and there is considerable evidence for it in the real world. This threshold is in addition to the threshold transitions at lower chemical potentials, from void to nuclear matter, and (presumably) from nuclear matter to two-flavor quark matter, as discussed above.

The second is that at equal chemical potential the fermi surfaces of the different quarks will not match. This mismatch cuts off the Cooper instability in mixed channels. If

the nominal gap is large compared to the mismatch, one can treat the mismatch as a perturbation. This will always be valid at asymptotically high densities, since the mismatch goes as m_s^2/μ , whereas the gap eventually grows with μ . If the nominal gap is small compared to the mismatch, condensation will not occur.

7.3. assembling the pieces

With these complications in mind, we can identify three major phases in the plane of chemical potential and strange quark mass, that reflect the simple microscopic physics we have surveyed above. (There might of course be additional “minor” phases – notably including normal nuclear matter!) There is 2-flavor quark matter, with restoration of chiral symmetry, and zero strangeness. Then there is a 2+1-flavor phase, in which the strange and non-strange fermi surfaces are badly mismatched, and one has independent dynamics for the corresponding low energy excitations. Here one expects strangeness to break spontaneously, by its own fermi surface instability. Finally there is the color-flavor locked phase. For some first attempts to sketch a global phase diagram, see [24] [25].

8. Comments

The recent progress, while remarkable, mainly concerns the asymptotic behavior of QCD. Its extrapolation to practical densities is at present semi-quantitative at best. To do real justice to the potential applications, we need to learn how to do more accurate analytical and numerical work at moderate densities.

As regards analytical work, we can take heart from some recent progress on the equation of state at high temperature [26] [27]. Here there are extensive, interesting numerical results [28], which indicate that the behavior is quasi-free, but that there are very significant quantitative corrections to free quark-gluon plasma results, especially for the pressure. Thus it is plausible *a priori* that some weak-coupling, but non-perturbative, approach will be workable, and this seems to be proving out. The encouraging feature here is that the analytical techniques used for high temperature appear to be capable of extension to finite density without great difficulty.

Numerical work at finite density, unfortunately, is plagued by poor convergence. This arises because the functional integral is not positive definite configuration by configuration, so that importance sampling fails, and one is left looking for a small residual from much larger canceling quantities.

There are cases in which this problem does not arise. It does not arise for two colors [29]. Although low-density hadronic matter is quite different in a two-color world than a three-color world – the baryons are bosons, so one does not get anything like a shell structure for nuclei – I see no reason to expect that the asymptotic, high-density phases should be markedly different. It would be quite interesting to see fermi-surface behavior arising for two colors at high density (especially, for the ground state pressure), and even more interesting to see the effect of diquark condensations.

Another possibility, that I have been discussing with David Kaplan, is to engineer lattice gauge theories whose low-energy excitations resemble those of finite density QCD near the fermi surface, but which are embedded in a theory that is globally particle-hole symmetric, and so feature a positive-definite functional integral.

Aside from these tough quantitative issues, there are a number of directions in which

the existing work should be expanded and generalized, that appear to be quite accessible. There is already a rich and important theory of the behavior of QCD at non-zero temperature and zero baryon number density. We should construct a unified picture of the phase structure as a function of both temperature and density; to make it fully illuminating, we should also allow at least the strange quark mass to vary. We should allow for the effect of electromagnetism (after all, this is largely what makes neutron stars what they are) and of rotation. We should consider other possibilities than a common chemical potential for all the quarks.

As physicists we should not, however, be satisfied with hoarding up formal, abstract knowledge. There are concrete experimental situations and astrophysical objects we must speak to. Hopefully, having mastered some of the basic vocabulary and grammar, we will soon be in a better position to participate in a two-way dialogue with Nature.

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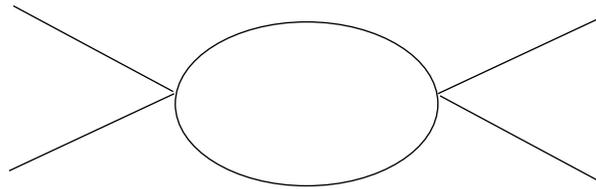


Figure 1. Graph contributing to the renormalization of four-fermion couplings.

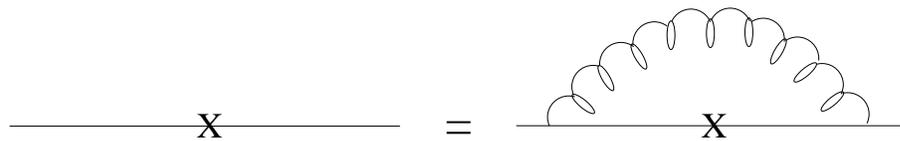


Figure 2. Graphical form of the self-consistent equation for the condensate (gap equation).